Applications of Diff^n Equation:-

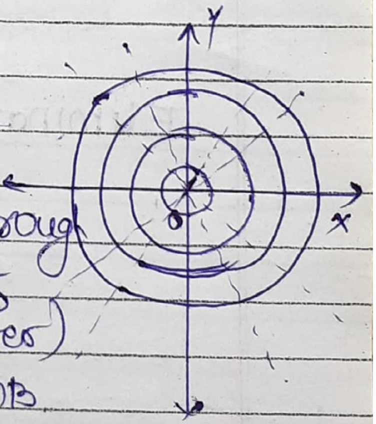
orthogonal Trajectories:-

① Trajectory:- A curve which cuts every member of a given family of curves according to some definite laws is called a trajectory of a family.

(2) orthogonal Trajectory:- A curve which cuts every member of a given family of curves at right angles is called as orthogonal trajectory of the family.

(3) orthogonal Trajectories:- Two families of curves are said to be orthogonal if every member of the either family cuts each member of the other family at right angles.

Thus, if the given family consist of straight lines $y = mx$ (m = const) representing family of straight lines all passing through the origin, then the family of circles $x^2+y^2=a^2$ (a is parameter) with centres at $(0,0)$ represents a family of orthogonal trajectories to the family $y = mx$.



Working Rule To find the equation of orthogonal Trajectories:-

(I) For rectangular cartesian co-ordinates:-

Step 1:- Given $F(x, y, a) = 0$, where a is a Variable parameter

Page No.
Date: / /

Step 2:- Differentiate $f(x,y,a)=0$ w.r.t x and eliminate 'a'. we thus form a differential equation of the family of the form
$$\phi\left(x,y,\frac{dy}{dx}\right)=0$$

Step 3:- Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$. Then the diffⁿ equation of the family of orthogonal trajectories will be $\phi\left(x,y,\frac{-dx}{dy}\right)=0$

Step 4:- The solution of step 3 is the family of orthogonal trajectories.

Que ① Find orthogonal trajectories of the family of parabolas $y=ax^2$.

Solⁿ:- we have $y=ax^2$ —(1) $\Rightarrow a=\frac{y}{x^2}$

Diffⁿ w.r.t x we get
$$\frac{dy}{dx}=2ax \quad \text{—(2)}$$

Eliminating 'a' from eqⁿ (1) & (2), we get
$$\frac{dy}{dx}=2\left(\frac{y}{x^2}\right)x$$
$$\therefore \frac{dy}{dx}=\frac{2y}{x} \quad \text{—(3)}$$

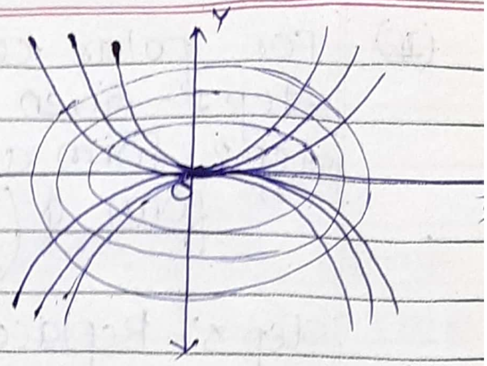
which is the diffⁿ eqⁿ for the family (1)

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in eqⁿ (3), we get
$$\therefore -\frac{dx}{dy}=\frac{2y}{x}$$
$$\Rightarrow x\,dx+2y\,dy=0 \quad \text{—(4)}$$

which is the diffⁿ equation of the orthogonal trajectories.

Page No.
Date.: / /

Integrating, (4), we get



$$\frac{x^2}{2} + y^2 = c^2$$

$$\Rightarrow \frac{x^2}{(\sqrt{2}c)^2} + \frac{y^2}{c^2} = 1$$

Which is the family of ellipse.

(2) Find the orthogonal trajectories of the curve given by $x^2 + 2y^2 = c^2$.

Soln:- Given $x^2 + 2y^2 = c^2$ —— (1)

diffn eqn (1) w.r.t. x

$$\Rightarrow 2x + 4y \frac{dy}{dx} = 0$$

$$\Rightarrow x + 2y \frac{dy}{dx} = 0 \quad \text{—— (2)}$$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (2),

we obtain diffn eqn of orthogonal trajectories as,

$$x + 2y\left(-\frac{dx}{dy}\right) = 0$$

$$\Rightarrow \frac{2\,dx}{x} = \frac{dy}{y}$$

Integrating, we get

$$2\log x = \log y + \log k$$

$$\Rightarrow x^2 = ky$$

$\therefore$ $x^2 = ky$ is the required orthogonal trajectories of (1).

Page No.
Date.: / /

(II) For polar co-ordinates:-

Step 1:- Given $F(r, \theta, a) = 0$, where a is paramet

Step 2:- form a diffⁿ eqⁿ of the family of the form $\phi\left(r, \theta, \frac{dr}{d\theta}\right) = 0$, by eliminating 'a'

Step 3:- Replace $\frac{dr}{d\theta}$ by $\left(-r^2 \frac{d\theta}{dr}\right)$ where by the diffⁿ eqⁿ of the family of orthogonal trajectorie become: $\phi\left(r, \theta, -r^2 \frac{d\theta}{dr}\right) = 0$.

Step 4:- Solve step-3 which is the family of orthogonal trajectories.

Que. ①:- Find the orthogonal trajectories of the circles defined by $r = a\cos\theta$ which all pass through the origin and have their centres on the initial line, a being the variable diameter.

Soln:

Given, $r = a\cos\theta$ —— (1)

$\log r = \log a + \log\cos\theta$.

∴ diffⁿ w.r.t θ, we get

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\cos\theta}\cdot(-\sin\theta)$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = -\tan\theta \quad \text{—— (2)}$$

which is the diffⁿ eqⁿ of the given family ①

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in ②, we get

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = -\tan\theta$$

$$\Rightarrow r\frac{d\theta}{dr} = \tan\theta$$

$$\therefore \frac{dr}{r} = \frac{d\theta}{\tan\theta} \Rightarrow \frac{dr}{r} = \cot\theta \cdot d\theta.$$

which is the diffⁿ eqⁿ of the family of ortho. traje-

Integrating, we get $\Rightarrow \log r = \log\sin\theta + \log c$.

$\Rightarrow \log r = \log(c + \sin\theta) \Rightarrow \boxed{r = c\sin\theta}$ required ortho. trajectories.

Newton's law of cooling:-

According to this law, the temperature of a body changes at a rate which is proportional to the difference in temperature between that of surrounding medium and that of body itself.

If θ_0 is the temp. of surroundings and θ that of the body at time t, then

$$\boxed{\frac{d\theta}{dt} = -k(\theta-\theta_0)}$$, where, k is const.

Ex:- ① A metal ball is heated to a temperature of $100°c$ & at time $t=0$ it is placed in a water which is maintained at $40°c$. If the temperature of the ball reduces to $60°c$ in 4 minutes, find the time at which the temperature of the ball is $50°c$.

Soln:-

Let the temperature of the ball be $T°c$ at time t minutes. Then By Newton's law of cooling,

$$\frac{dT}{dt} = -k(T-T_0) \quad \text{---(1)}$$

Given $T_0 = 40°c$ is the temperature of water

i.e. $dT/dt = -k(T-40)$

$$\therefore \frac{dT}{(T-40)} = -k\,dt.$$

Integrating,

$$\int \frac{dT}{(T-40)} = -k\int dt + c.$$

$$\Rightarrow \log(T-40) = -kt + \log c \quad \text{---(2)}$$

$$\Rightarrow -kt = \log\frac{(T-40)}{c} \quad \text{---(3)}$$

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Page No.
Date: / /

Here we have two unknowns k and c.
To find these two unknowns two conditions are given,

At $t=0$, $T=100$.

Substituting in eqⁿ (2), we get

$$\log c = \log 60$$
$$\therefore \boxed{c=60}$$

Thus substituting value of c in eqⁿ (3)

$$-kt = \log \frac{T-40}{60} \quad \text{----(4)}$$

Also, at $t=4$, $T=60$ substituting in eqⁿ (4)

$$-4k = \log \frac{1}{3} \Rightarrow k = \frac{1}{4}\log 3$$

substituting value of k in eqⁿ (4)

$$-\frac{t}{4}\log 3 = \log \frac{T-40}{60} \quad \text{----(5)}$$

is the relation in t & T.

Now, to find the time at $T=50$
substituting $T=50$ in eqⁿ (5)

$$t = \frac{4\log 6}{\log 3} = 6.5 \text{ minutes}$$

(2) A body originally at 80°c cools down to 60°c in 20 minutes, the temperature of the air being 40°c. What will be the temperature of the body after 40 minutes from the original?

Soln:- Let θ be the temperature of a body at any time 't'.
By Newton's law of cooling,

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Date
Page

$$\frac{d\theta}{dt} = -k(\theta - 40) \qquad (\because \theta_0 = 40), k\text{-cons}$$

$$\therefore \frac{d\theta}{\theta - 40} = -k\,dt \qquad \text{--- (1)}$$

At $t=0$, $\theta = 80°c$ & when $t=20$, $\theta = 60°c$

Integrating eqⁿ (1) with these limits, we g

$$\int_{80}^{60} \frac{d\theta}{\theta - 40} = -k \int_{0}^{20} dt$$

$$\log(\theta - 40)\Big]_{80}^{60} = -k[t]_0^{20}$$

$$\log 20 - \log 40 = -20k$$

$$\Rightarrow \log\frac{1}{2} = -20k$$

$$\Rightarrow k = \frac{1}{20}\log 2 \qquad \text{--- (2)}$$

let $\theta = T$ when $t = 40$. But $\theta = 80$ when $t=0$

$\therefore$ Equation (1) becomes

$$\int_{80}^{T} \frac{d\theta}{\theta - 40} = -k\int_{0}^{40} dt$$

$$\therefore \log\left(\frac{T-40}{40}\right) = -40k$$

$$\Rightarrow \log\left(\frac{T-40}{40}\right) = -40\left(\frac{1}{20}\log 2\right) = -2\log 2 = \log\frac{1}{4}$$

$$\Rightarrow \frac{T-40}{40} = \frac{1}{4}$$

$$\Rightarrow T - 40 = 10$$

$$\Rightarrow T = 50°c.$$

(3) A body of temperature 100°c is placed in a room whose temperature is 20°c & cool to 60°c in 5 minutes. What will be its temp. after a further interval of time ...

Date

Page

Soln:- Let θ be the temperature of body of any time 't' and θ_0 be the temperature of surrounding medium.

$$\theta_0 = 20^\circ c.$$

By Newton's law of cooling

$$\frac{d\theta}{dt} = -k(\theta - \theta_0)$$

$$\frac{d\theta}{dt} = -k(\theta - 20)$$

$$\Rightarrow \frac{d\theta}{\theta - 20} = -k\,dt \quad \text{--- (1)}$$

We have, in 5 minutes body cools down from $100^\circ c$ to $50^\circ c$.

i.e. at $t = 0$, $\theta = 100^\circ c$

at $t = 5$, $\theta = 60^\circ c$.

$$\therefore \int_{100}^{60} \frac{d\theta}{\theta - 20} = -k \int_0^5 dt$$

$$\text{i.e. } \left[\log(\theta - 20)\right]_{100}^{60} = -k\,[t]_0^5$$

$$\Rightarrow \log 40 - \log 80 = -5k$$

$$\text{i.e. } k = \frac{1}{5} \log \frac{80}{40} = \frac{1}{5} \log 2 \quad \text{--- (2)}$$

Next, let in 10 minutes body cools from $100^\circ c$ to $\theta^\circ c$ (say)

$\therefore$ Equation (1) becomes.

$$\int_{100}^{\theta} \frac{d\theta}{\theta - 20} = -k \int_0^{10} dt$$

$$\log(\theta - 20) - \log 80 = -10k = -10\left(\frac{1}{5} \log 2\right)$$

$$\therefore \log\left(\frac{\theta - 20}{80}\right) = -2\log 2 = \log \frac{1}{4}$$

$$\therefore \frac{\theta - 20}{80} = \frac{1}{4}$$

$$\therefore \theta - 20 = 20$$

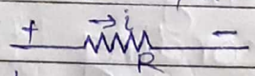
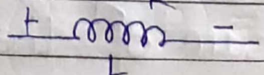
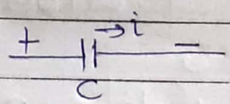
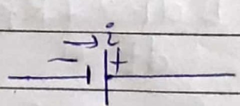
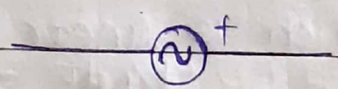
$$\Rightarrow \theta = 40^\circ C$$

Simple Electric circuits :-

- Circuits made up of-
(i) three passive elements :- A passive elements transform or store energy but not an energy source.
  e.g. resistance, inductance, capacitance.

(ii) an active elements :- An energy source is voltage source or a current is said to be active elements.
  e.g. Battery or generator.

(1) Table of Elements, symbols and units :-

Elements	symbol	unit
• Time	t	second.
• Quality of electricity.	$q = \int i\,dt$	coulomb.
• current	$i = dq/dt$	ampere (A)
• Resistance, R		ohm (Ω)
• Inductance, L		henry (H)
• Capacitance, C		Farad (F)
• Electromotive force or voltage (const) E	 Battery E = const.	Volts.
• Variable voltage generator	 Generator, E = Variable Voltage	volt.

Date
Page

(2) **Basic Relations:-**

(i) $i = \frac{dq}{dt}$ or $q = \int i\,dt$ ($\because$ current is the rate of flow of electricity)

(ii) voltage drop across resistance $R = Ri$ (ohm's law)

(iii) voltage drop across inductance $L = L\frac{di}{dt}$

(iv) voltage drop across capacitance $C = \frac{q}{C}$

(3) Kirchhoff's law:- The formulation of diffn eqns for an electrical circuit depends on the following two Kirchhoff's law which are of cardinal importance:-

- (I) The algebraic sum of the voltage drops around any closed circuit is equal to the resultant electromotive force in the circuit.
- (II) The algebraic sum of the currents flowing into any node is zero.

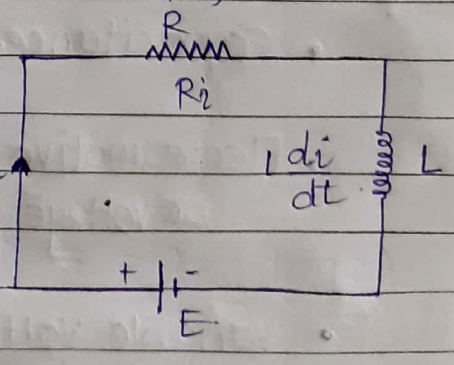
(4) Differential Equations:-

(i) circuit involving L and R along with a voltage source (battery E), all in series:-

If R, L and voltage source E are connected in series and switch is closed, i is current flowing through a circuit, then.

According to Kirchoff's law of voltage:

sum of the voltage drops across R & L = Total e.m.f across R & L.



i.e. $L\frac{di}{dt} + Ri = E$.

$\Rightarrow \frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$.

which is linear diffn eqn.

$\therefore$ I.F $= e^{\int R/L\,dt} = e^{R/L\,t}$.

& G S is $i(I\cdot F) = \int \frac{E}{L}(I\cdot F)dt + k$

$$\Rightarrow i e^{R/L\, t} = \int \frac{E}{L} e^{(R/L)t} dt + k$$

$$\Rightarrow i e^{R/L\, t} = \frac{E}{L} \frac{e^{(R/L)t}}{R/L} + k = \frac{E}{L} \cdot \frac{L}{R} e^{Rt/L} + k$$

$$\Rightarrow i = \frac{E}{R} + k e^{-Rt/L}$$ which gives current at any time 't' — (1)

If initially there is no current in the circuit i.e $i=0$, when $t=0$, then we have,

$$k = -\frac{E}{R}$$

Thus eqⁿ (1) becomes $i = \frac{E}{R}\left(1 - e^{-Rt/L}\right)$.

As $t \to \infty$, $i = \frac{E}{R}$, which shows that i increases with t and attains the maximum value E/R.

(ii) circuits involving R and C along with a voltage source (battery) E all in series:-

Consider a circuit containing resistance and capacitance C in series with a voltage source (battery) E.

Let i be the current flowing in the circuit at any time t. Then by Kirchhoff's first law, we have

sum of voltage drop across R & C = E

i.e. $Ri + \frac{q}{C} = E$.

since, $i = \frac{dq}{dt}$, this equation in terms of q can be written as,

$$R\frac{dq}{dt} + \frac{q}{C} = E \Rightarrow \frac{dq}{dt} + \frac{q}{RC} = E$$

which is linear diffⁿ equation.

$$I.F = e^{\int \frac{1}{RC} dt} = e^{t/RC}$$

$\therefore$ soln is,

$$q \cdot e^{t/RC} = \int \frac{E}{R} e^{t/RC} dt + B$$

$$= \frac{E}{R} \cdot \left(\frac{e^{t/RC}}{1/RC} \right) + B$$

$$= \frac{E}{R} \left(RC \cdot e^{t/RC} \right) + B.$$

$$q = EC + Be^{-t/RC} \quad \text{—— (1)}$$

Assuming

$q = q_0$ when $t = 0$

(1) $\Rightarrow$

$q_0 = EC + B$ giving $B = q_0 - EC$

$$\therefore q = EC + (q_0 - EC) e^{-t/RC}$$

$$q = EC\left(1 - e^{-t/RC}\right) + q_0 e^{-t/RC}.$$

$$i = \frac{dq}{dt} = EC \frac{1}{RC} e^{-t/RC} - \frac{q_0}{RC} e^{-t/RC}.$$

$$\therefore i = \left(\frac{E}{R} - \frac{q_0}{RC} \right) e^{-t/RC}$$

(iii) Circuit involving L and C both in series, after Removing source applied e.m.f :-

consider a circuit containing inductance L & capacitance C in series without e.m.f.

let i be the current flowing in the circuit at any time t. Then by Kirchoff's first law, we have,

sum of Voltage drop across L & C $= 0$.

Date
Page

i.e. $L\frac{di}{dt} + \frac{q}{C} = 0$

$\therefore \frac{di}{dt} = \frac{-q}{LC} \Rightarrow \frac{di}{dq}.\frac{dq}{dt} = \frac{-q}{LC}$

$\Rightarrow i\frac{di}{dq} = \frac{-q}{LC}$

$$\int i\,di = -\int \frac{q}{LC}\,dq + A$$

$$\Rightarrow \frac{i^2}{2} = -\frac{q^2}{2LC} + A$$

$$\Rightarrow i^2 = \frac{-q^2}{LC} + B$$

Assuming $i=0$, $q=q_0$, when $t=0$

$$\therefore B = \frac{q_0^2}{LC}$$

$$\Rightarrow i^2 = \frac{(q_0^2 - q^2)}{LC} \Rightarrow i = \pm\frac{\sqrt{(q_0^2-q^2)}}{\sqrt{LC}}$$

since q decreases as t increases,

$$i = \frac{dq}{dt} = -\frac{1}{\sqrt{LC}}\sqrt{q_0^2 - q^2}$$

$$\frac{-dq}{\sqrt{q_0^2-q^2}} = \frac{dt}{\sqrt{LC}} \Rightarrow \cos^{-1}\left(\frac{q}{q_0}\right) = \frac{t}{\sqrt{LC}} + c$$

$\because q = q_0$, when $t=0$, $\therefore c=0$

$$\therefore \Rightarrow \frac{t}{\sqrt{LC}} = \cos^{-1}\left(\frac{q}{q_0}\right) \Rightarrow \boxed{q = q_0 \cos\left(\frac{t}{\sqrt{LC}}\right)}$$

Useful formulae:-

(1) $\int e^{at}\sin bt\,dt = \frac{e^{at}}{a^2+b^2}(a\sin bt - b\cos bt)$

(2) $\int e^{at}\cos bt\,dt = \frac{e^{at}}{a^2+b^2}(a\cos bt + b\sin bt)$

(3) $\int e^{at}\sin bt\,dt = \frac{e^{at}}{\sqrt{a^2+b^2}}\sin(bt-\phi)$, where $\phi = \tan^{-1}\left(\frac{b}{a}\right)$

(4) $\int e^{at}\cos bt\,dt = \frac{e^{at}}{\sqrt{a^2+b^2}}\cos(bt-\phi)$, where $\phi = \tan^{-1}($

Date
Page

EX. (1) A Resistance of 100 ohms, an inductance of 0.5 henry are connected in series with a battery of 20 volts. find the current in a circuit as a function of t.

Sol^n:- By Kirchhoff's law, we have.

$$L\frac{dI}{dt} + RI = E$$

$$\frac{dI}{dt} + \frac{R}{L} I = \frac{E}{L} \quad \text{which is linear.}$$

Here $P = \frac{R}{L}$, $Q = \frac{E}{L}$

$$\therefore I.F = e^{\int \frac{R}{L} dt} = e^{Rt/L}$$

$\therefore$ G.S is, $I \cdot (I.F) = \int Q \cdot (I.F)\, dt + A$

$$\Rightarrow I \cdot e^{Rt/L} = \int \frac{E}{L} \cdot e^{Rt/L} dt + A$$

$$= \frac{E}{\cancel{L}} \cdot \frac{\cancel{L}}{R} \cdot e^{Rt/L} + A.$$

$$I \cdot e^{Rt/L} = \frac{E}{R} e^{Rt/L} + A. \quad \dots (1)$$

But at $t=0$, $I=0$

$$\Rightarrow 0 = \frac{E}{R} + A. \quad \Rightarrow A = -\frac{E}{R}$$

$\therefore$ G.S becomes,

$$I \cdot e^{Rt/L} = \frac{E}{R}\left(-1 + e^{Rt/L}\right)$$

$$\Rightarrow I = \frac{E}{R}\left(1 - e^{-Rt/L}\right)$$

Given that $R = 100$ ohms, $L = 0.5$ Henry.
$E = 20$ volts.

$$\Rightarrow I = \frac{20}{100}\left(1 - e^{-\frac{100t}{0.5}}\right) = \frac{1}{5}\left(1 - e^{-200t}\right)$$

EX. (2) Find the current i in the circuit having resistance R and condenser of capacity c in series with e.m.f $E\sin wt$.

Sol^n:- By Kirchoff's law,

sum of potential drops = Total e.m.f. applied.

$$\Rightarrow Ri + \frac{q}{C} = E\sin wt$$

$$\Rightarrow R\frac{dq}{dt} + \frac{q}{C} = E\sin wt$$

$$\frac{dq}{dt} + \frac{q}{RC} = \frac{E\sin wt}{R}$$ which is linear diff^n eq^n in q.

$$\therefore I.F = e^{\int \frac{1}{RC} dt} = e^{t/RC}$$

$\therefore$ G.S is $\Rightarrow$

$$q.(I.F) = \int Q.(I.F)\,dt + A.$$

$$q.e^{t/RC} = \int \frac{E}{R}\sin wt \,.\, e^{t/RC}\,dt + A.$$

$$= \frac{E}{R}\cdot\frac{e^{t/RC}}{\sqrt{1/R^2C^2 + w^2}}\sin(wt - \phi) + A.$$

, where $\tan\phi = RCw$.

$$= \frac{EC \,.\, e^{t/RC}}{\sqrt{1 + R^2C^2w^2}}\sin(wt - \phi) + A.$$

$$\therefore q = \frac{EC}{\sqrt{1+R^2C^2w^2}}\sin(wt - \phi) + A.e^{-t/RC}.$$

$$\therefore i = \frac{dq}{dt} \Rightarrow \frac{ECw}{\sqrt{1+R^2C^2w^2}}\cos(wt - \phi) - \frac{A}{RC}.e^{-t/RC}.$$

Date
Page

Rectilinear Motion:-

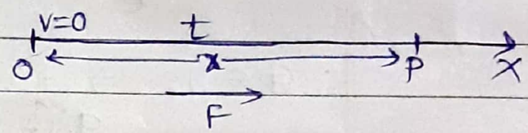
Rectilinear motion is a motion of a body along a straight line. Let a body m start moving from a fixed point O along a straight line ox under the action of a force F. After any time t, let it be moving at p where op = x, then

(i) its velocity (v) $= \frac{dx}{dt}$

(ii) its acceleration (a) $= \frac{dv}{dt} \Rightarrow \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$

also, $a = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = \frac{dv}{dx} \cdot v$

$\therefore a = v \cdot \frac{dv}{dx}$



$\therefore$ acceleration $= a = \frac{dv}{dt}$ or $v \cdot \frac{dv}{dx}$ or $\frac{d^2x}{dt^2}$.

Newton's second law of motion:-

It states that the rate of change of momentum (mass X velocity) at a body is proportional to the resultant force acting on a body.

If 'm' is const. then Newton's second law is,

$$F = m\frac{dv}{dt} = ma = \text{Net force}.$$

$\therefore F = m \cdot \frac{dv}{dt}$ or $mv\frac{dv}{dx}$ or $m\frac{d^2x}{dt^2}$, where, F is th effective forc

D'Alembert's principle:- Algebraic sum of the force acting on a body along a given direction is equal to the product of mass and acceleratio in that direction.

i.e. $\boxed{\text{Net force} = \text{mass} \times \text{Acceleration}}$

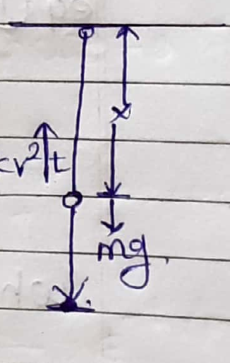
Note:- The forces usually are:
(i) vertically downward (ii) tension in elastic string or sping, (iii) reactions or stresses at point in contact with other bodies, (iv) Forces of attraction (v) forces of resistance due to wind & friction etc.

EX. (1) A body of mass m, falling from rest is subjected to the force of gravity and an air resistance proportional to the square of the velocity (kv^2). If it falls through a distance x and possesses a velocity v at that instant, prove that $\frac{2kx}{m} = \log\left(\frac{a^2}{a^2-v^2}\right)$ where $mg = ka^2$.

Sol^n- The forces acting on the body are:
(i) The gravitational accel^n i.e. the weight acting downwards $= mg$ (vertically downward)
(ii) the air resistance acting upwards $= -kv^2$.

$\therefore$ Net force on the body $= mg - kv^2$
$\therefore$ By Newton's second law, the equation of motion is,

$$mv\frac{dv}{dx} = mg - kv^2$$



$$\therefore mv\frac{dv}{dx} = ka^2 - kv^2$$

$$(\because mg = ka^2)$$

$$mv\frac{dv}{dx} = k(a^2 - v^2)$$

$$\Rightarrow \int \frac{v\,dv}{a^2-v^2} = \int \frac{k}{m}dx$$

$$-\frac{1}{2}\log(a^2-v^2) = \frac{kx}{m} + c \quad \text{——— (1)}$$

Initially $x = 0$, when $v = 0$.
$\therefore$ from eq^n (1) $\boxed{c = -\frac{1}{2}\log a^2}$

Date
Page

$\therefore \frac{1}{2}\left[\log a^2 - \log(a^2 - v^2)\right] = \frac{kx}{m}$

$\Rightarrow \frac{2kx}{m} = \log\left(\frac{a^2}{a^2-v^2}\right)$

EX. (2) A body starts moving from rest is opposed by a force per unit mass of value cx & resistance per unit mass of value bv^2. where x and v are the displacement & velocity of the body at that instant. show that the velocity of the body is given by.

$$v^2 = \frac{c}{2b^2}\left(1 - e^{-2bx}\right) - \frac{cx}{b}$$

Solⁿ: By Newton's second law of motion, the equation of motion of the body is

$$v\frac{dv}{dx} = -cx - bv^2$$

$$\Rightarrow v\frac{dv}{dx} + bv^2 = -cx \quad \text{--- (1)}$$

This is Bernoulli's eqⁿ, put $v^2 = z$

$$\Rightarrow 2v\frac{dv}{dx} = \frac{dz}{dx}$$

eqⁿ (1) becomes,

$$\frac{1}{2}\frac{dz}{dx} + bz = -cx$$

$$\Rightarrow \frac{dz}{dx} + 2bz = -2cx$$

which is linear equation &

$$I.F = e^{\int 2b\,dx} = e^{2bx}$$

$\therefore$ G.S is,

$$z \cdot e^{2bx} = \int -2cx \cdot e^{2bx}\,dx + c_1$$

$$= -2c\int x \cdot e^{2bx}\,dx + c_1$$

$$z \cdot e^{2bx} = -\cancel{2}c\left[x \cdot \frac{e^{2bx}}{\cancel{2}b} - \int 1 \cdot \frac{e^{2bx}}{2b}\, dx\right] + c_1$$

$$\therefore v^2 e^{2bx} = -\frac{cx}{b} e^{2bx} + \frac{c}{2b^2} e^{2bx} + c_1$$

$$v^2 = \frac{-cx}{b} + \frac{c}{2b^2} + c_1 e^{-2bx} \quad \text{——— (2)}$$

Initially when $x=0$, $v=0$

$$\therefore \frac{c}{2b^2} + c_1 = 0 \Rightarrow c_1 = \frac{-c}{2b^2}.$$

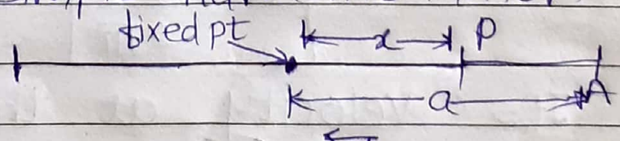
substituting the value of c_1 in (2), we have

$$v^2 = -\frac{cx}{b} + \frac{c}{2b^2} - \frac{c}{2b^2} e^{-2bx}$$

$$v^2 = \frac{c}{2b^2}\left(1 - e^{-2bx}\right) - \frac{cx}{b}$$

Simple Harmonic Motion :-

If a particle moves on a straight line so that the force acting on it is always directed towards a fixed point on the line and proportional to its distance from the point, the particle is said to move in simple Harmonic Motion.



Force = $m\mu x$.

(I) Let O be the fixed pt. & P be the position of particle at any time t.
Let $OP = x$. Force acting on particle is $m\mu x$, where m is mass of the particle & μ a const. considered positive.

Date
Page

Equation of motion is,

$$m\frac{d^2x}{dt^2} = -m\mu x$$

$$\Rightarrow v\frac{dv}{dx} = -\mu x.$$

$$v\,dv = -\mu x\,dx.$$

Integrating, we get

$$v^2 = -\mu x^2 + A.$$

Assuming that particle starts from point A (OA=a) & its initial velocity is zero when $x=a$, $v=0$

$$\therefore A = \mu a^2$$

$$\therefore v^2 = \mu(a^2-x^2) \Rightarrow v = \pm\sqrt{\mu}\sqrt{a^2-x^2}$$

or $$v = \frac{dx}{dt} = -\sqrt{\mu}\sqrt{a^2-x^2}$$

Negative sign is attached because as t increases x decreases.

$$-\frac{dx}{\sqrt{a^2-x^2}} = \sqrt{\mu}\,dt.$$

Integrating, $$\cos^{-1}\frac{x}{a} = \sqrt{\mu}t + B$$

when $t=0$, $x=a$ $\therefore B=0$

we get, $$x = a\cos(\sqrt{\mu}\cdot t)$$

particle will reach O in time t_1, given by

$$0 = a\cos(\sqrt{\mu}\cdot t_1)$$

$$\therefore \boxed{t_1 = \frac{\pi}{2\sqrt{\mu}}}$$

Its velocity at that time will be $\sqrt{\mu}a$. As soon as it will cross O, the direction of force will change, however, particle will move with velocity $\sqrt{\mu}a$ & ultimately come to rest at OA=OA' & time taken by the particle to travel from O & A' will be $\frac{\pi}{2\sqrt{\mu}}$.

Due to attraction, particle will start moving towards O. It is oscillatory motion. Due to this it is called S.H.M. period of oscillation is $\frac{2\pi}{\sqrt{\mu}}$.

(II) Hook's Law:-

suppose that an elastic string of negligible weight hangs vertically with its upper end fixed with a mass of M kg attached to its lower end. Let l meters be the natural length of the spring and let T be its tension when it is streched to a length x meters. Then the extension of the string beyond its natural length is given by $(x-l)/l$. The tension and the extension are related by Hooks law:-

The tension of an elastic string or spring is proportional to the extension of the spring beyond its natural length. we thus have:

$$T = \frac{\lambda}{l}(x-l)$$

where λ is a const. called the modulus of elasticity of the string. The value λ depends on the material. Then the tension is λx.

The acceln of the mass is $\frac{dv}{dt}$ or $v.\frac{dv}{dx}$ so that the eqn of motion written as,

$$\cancel{M}v\frac{dv}{dx} = mg - \lambda x.$$

$$\Rightarrow \quad v\,dv = \left(g - \frac{\lambda}{M}x\right)dx.$$

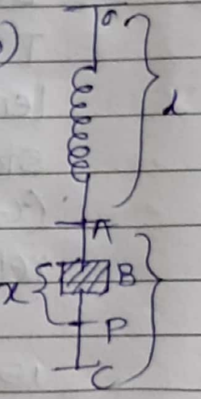
Integrating, $\frac{v^2}{2} = gx - \frac{\lambda}{M}\frac{x^2}{2} + c$ —— (1)

To calculate c, we assume that $v = v_0$ when $x=0$. This gives $c = v_0^2/2$, eqn (1) becomes

$$\frac{v^2}{2} = gx - \frac{\lambda}{M}\frac{x^2}{2} + \frac{v_0^2}{2} \Rightarrow \boxed{v^2 = 2gx - \frac{\lambda}{M}x^2 + v_0^2}$$

(III) Motion of a particle of mass m suspended by an Elastic string :-

Let OA be an elastic string of natural length 'l' with fixed end 'O'. Let a mass 'm' be attached to 'A' so that the string elongate and the mass 'm' reaches a position of equilibrium at B. consider the equillibrium of the mass 'm' at position of equillibrium at B. It is acted on by force 'mg' downward due to gravity & tension 'T' upwards due to extension. In equillibrium.



$$mg = T = \frac{\lambda AB}{l} \text{ (by Hook's Law)}$$

If $AB = c$ then $\frac{mgl}{\lambda} = c$ ...... ①

Now, let the particle be pulled down to 'C' & then let go.

Let $BC = d$. The mass will go up due to tension in the spring.

Let the mass be at 'P' between B & C and $BP = x$.

The force acting on the mass is $mg - T$ downwards.

$\therefore$ Eqn of motion $m \cdot \frac{d^2x}{dt^2} = mg - T$

$$= mg - \frac{\lambda \times \text{Extension}}{\text{Natural length}}$$

$$= mg - \frac{\lambda (AB + x)}{l}$$

$$= mg - \frac{\lambda}{l}\left(\frac{mgl}{\lambda} + x\right) \text{ by ①}$$

$$= -\frac{\lambda x}{l}$$

$$\therefore \boxed{\frac{d^2x}{dt^2} = -\frac{\lambda}{lm} x}$$

This equation is same as that of S.H.M except $\mu = \lambda/m$.

Page No.
Date: / /

Hence the particle of mass m will execute S.H.M. of period $\frac{2\pi}{\sqrt{\frac{\lambda}{lm}}}$. The expression $\frac{mgl}{\lambda}$ is called static extension of the spring & denoted by c. Under this case, period of oscillation $2\pi\sqrt{\frac{lm}{\lambda}} = 2\pi\sqrt{\frac{c}{g}}$.

$\therefore$ period $= 2\pi\sqrt{\frac{\text{static extension}}{g}}$

Importance of c is obvious if period of oscillation is compared with that of simple pendulum of length l. i.e. period of oscillation of mass is that of simple pendulum of length $\frac{lmg}{\lambda}$ i.e. static extension of spring.

Page No.
Date: / /

Ex:- An elastic spring of natural length l is fixed at a point A. To the lower end is attached a particle of mass m so that the spring stretches to a length $2l$. If the particle is dropped from A, show that it descends a distance $l(2+\sqrt{3})$ before coming to rest.

Solⁿ:-

The tension in the spring is given by

$$T = \frac{\lambda(2l-l)}{l} = \lambda. \quad \text{--- (1)}$$

Since this balances the weight of the particle, we have $\lambda = mg$.

At time t, let x be the extension of the spring beyond its natural length. Then the eqⁿ of motion of the particle is given by

$$m \cdot v\frac{dv}{dx} = mg - T$$

$$= mg - \frac{\lambda x}{l} = mg - mg \cdot \frac{x}{l} = mg\left(1 - \frac{x}{l}\right)$$

$$\therefore v\,dv = g\left(1 - \frac{x}{l}\right)dx. \quad \text{--- (2)}$$

Integrating, $\frac{v^2}{2} = gx - \frac{gx^2}{2l} + c$ --- (3)

Since the particle is dropped from A, we have $v = \sqrt{2gl}$, when $x = 0$.

Substituting these values in (3) we find $c = lg$.

$\therefore$ eqⁿ (3) becomes, $\frac{v^2}{2} = gx - \frac{g x^2}{2l} + gl$

When $v = 0$, x is given by $x^2 - 2lx - 2l^2 = 0$.

from which we obtain. $x = l + l\sqrt{3}$

Hence, the distance by which the particle descends before coming to rest is given by

$x + l = 2l + l\sqrt{3} = l(2+\sqrt{3})$

Heat flow:-

The fundamental principles involved in the problems of heat conduction are

(i) Heat flows from a higher temp. to the lower temp.

(ii) The quantity of heat in a body is proportio to its mass & temp.

(iii) Fourier law of Heat conduction:-

The rate of heat flow across an area is proportional to the area and to the rate of change of temp. w.r.t. its distance normal to the area.

If q (cal/sec) be the quantity of heat that flows across a slab of area A (cm^2) & thickness δx in one second. where the diffn of temp. at the faces is δT. then by (iii) above,

q = Thermal Conductivity × Area × Temp gradient

$$\boxed{q = -KA\frac{dT}{dx}}$$

where K is a const depending upon the material of the body is called the thermal conductivity.

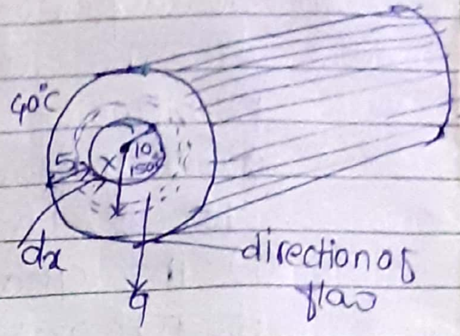
— Negative sign attached because T decreases as x increases.

EX① A pipe 20 cm in diameter contains steam at 150°C & is protected with a covering 5cm thick for which K=0.0025. If the temp of the outer surface of the covering is 40°c, find the temp. half-way through the covering under steady-state conditions.

Soln.-

Let q cal/sec be the constant quality of heat flowing out radially through a surface of the pipe having radius x cm & length 1 cm

Then the area of the lateral surface $= 2\pi x$



Hence by fourier law,

$$q = -KA\frac{dT}{dx}$$

$$= k \cdot 2\pi x \frac{dT}{dx}$$

$$dT = \frac{-q}{2\pi k} \cdot \frac{dx}{x}$$

Integrating,

$$T = \frac{-q}{2\pi k} \log_e x + c$$

since $T = 150$, when $x = 10$.

$$\therefore 150 = \frac{-q}{2\pi k} \log_e 10 + c \quad \text{---} (1)$$

& $T = 40$ when $x = 15$

$$\therefore 40 = \frac{-q}{2\pi k} \log_e 15 + c. \quad \text{---} (2)$$

subtracting (2) from (1)

$$110 = \frac{q}{2\pi k} \log_e 1.5 \quad \cdots (3)$$

Let $T = t$, when $x = 12.5$

$$\therefore t = \frac{-q}{2\pi k} \log_e 12.5 + c. \quad \cdots (4)$$

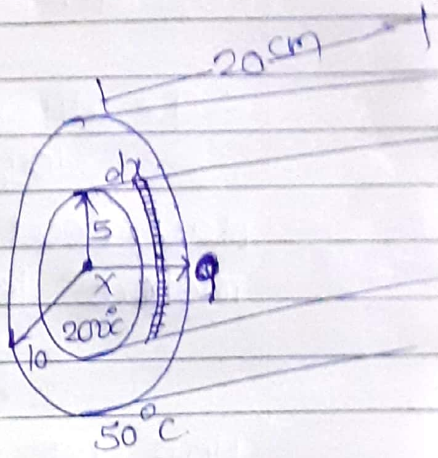
subtracting (1) from (4),

$$t - 150 = \frac{-q}{2\pi k} \log_e 1.25 \quad \text{---} (5)$$

dividing (5) by (3)

$$\frac{t-150}{110} = -\frac{\log_e 1.25}{\log_e 1.5} \Rightarrow \boxed{t = 89.5°C}$$

(2) A long hollow pipe has an inner diameter of 10cm & outer diameter of 20cm. The inner surface is kept at 200°c & the outer surface at 50°c. The thermal conductivity is 0.12. How much heat is lost per minute from a portion of pipe 20 meters long? Find the temp at a distance $x = 7.5$ cm from the centre of the pipe.



Solⁿ:- consider one ~~___~~ cylinder of radius x cm & length 1cm. The surface area of this cylinder is $A = 2\pi x$. Let q be the quality of heat flowing across this surface, then.

$$q = -KA\frac{dT}{dx}$$

$$q = -k \cdot 2\pi x \frac{dT}{dx}$$

$$\Rightarrow dT = \frac{-q}{2\pi K} \cdot \frac{dx}{x}$$

Integrating, $T = \frac{-q}{2\pi K} \log_e x + c$. ——— (1)

since $T = 200$ when $x = 5$

$$\therefore 200 = \frac{-q}{2\pi K} \log_e 5 + c \quad \text{——— (2)}$$

Also $T = 50$, when $x = 10$

$$\therefore 50 = \frac{-q}{2\pi K} \log_e 10 + c \quad \text{——— (3)}$$

Subtracting (3) from (2), we get

$$150 = \frac{-q}{2\pi K} (\log_e 10 - \log_e 5)$$

$150 = \frac{q}{2\pi K} \log_e 2$ —— (4)

$\therefore q = \frac{2\pi K \times 150}{\log_e 2} = \frac{300\pi \times 0.12}{\log_e 2} = 163$ cal/sec.

Now, heat lost per minute through 20 metre length of pipe.

$= 60 \times 2000 \times q$

$= 120000 \times 163 = 1956000$ cal.

Now, let $T = t$, when $x = 7.5$

$\therefore$ from (1)

$$t = \frac{-q}{2\pi K} \log_e 7.5 + C \quad \text{—— (5)}$$

Subtracting (2) from (5), we get.

$$t - 200 = \frac{-q}{2\pi K} (\log_e 7.5 - \log_e 5)$$

$$t - 200 = \frac{-q}{2\pi K} \log_e 1.5 \quad \text{—— (6)}$$

Dividing (6) by (4), we get

$$\frac{t - 200}{150} = -\frac{\log_e 1.5}{\log_e 2}$$

$\Rightarrow t = 200 - 150 \times 0.58 = 113$

$\therefore$ when $x = 7.5$ cm, $\boxed{T = 113^\circ C}$